Mathematical Statistics Qualifier Examination
Part I of the STAT AREA EXAM
May 26, 2021; 9:00 AM - 11:00 AM

There are 4 problems. You are required to solve them all. Show detailed work for full credit. You need to turn in your exam by 11:05 am, and receive the questions for your applied statistics exam at 11:15 after a break.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

NAME: ______________________________ **ID:** ______________________________

Signature: ______________________________

Instructions:

- This exam is conducted **via Zoom on May 26 from 9:00 to 11:00 am EDT.**
- The entire Zoom meeting and chat messages are being recorded.
- This is a closed book, closed note exam.
- Hand calculators (or other computing devices) may not be used during the exam.
- You should join the Zoom meeting from two devices: Your computer/laptop/tablet (with webcam), and your smartphone (with camera).
- Audio should be muted and video must be kept on during the exam.
- Your computer webcam must fully show your face; your smartphone camera should show your computer monitor, your hands and workspace, with the pages of paper being used for the exam.
- The gallery view must be kept off.
- At the very beginning of the exam, during set up, you will be asked to do a brief "environment scan", showing the workspace where your computer is and the desk/table/floor where you will be writing your work.
- You are required to bring enough blank pieces of paper to use for the exam. You will show the blank pages at the beginning, during the "environment scan" on Zoom.
- You are not allowed to use the internet for any searching or communication with others, with the sole exception of communication with the proctors via Zoom chat (which is set so that your chats only go privately to hosts, not to others).
- It is recommended that you print the exam and write your answers on it. However, you can write your answers on your blank papers if you do not have a printer with you.
- After you finish the exam, scan your pages, ordered and oriented appropriately, into a single pdf file. Email the pdf file to **hongshik.ahn@stonybrook.edu** no later than 5 minutes after completion of the exam (i.e., **by 11:05 am EDT**).
- No students are allowed to leave the Zoom meeting until the exam is over.
  - If you finish the exam early, then submit your exam and remain in the Zoom meeting until the conclusion of the exam at 11:00 am EDT.
  - After submitting your exam, you can study for another exam or work on anything else, while staying in view in the Zoom meeting.
- If the answers are not submitted by 11:05 am EDT, the exam will not be graded, and a score of zero will be given.
- If you have a question during the exam, then send a chat message to the host privately.

1. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f(x) = \begin{cases} 1/\theta & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathrm{E}X_{(1)}$ and $\mathrm{E}X_{(n)}$.

(b) Find $\mathrm{E}\left(\dfrac{X_{(1)}}{X_{(n)}}\right)$.

2. Let $X_1, X_2, \cdots$ be iid with the uniform distribution on $[0, 1]$. Find the number a so that

$$\sqrt{n}\left[\left(\prod_{i=1}^{n} X_i\right)^{1/n} - a\right]$$

converges in distribution, and identify the limiting distribution. (Hint: Define $Y_i = \log X_i$, $i = 1, \ldots, n$ and apply the CLT on $Y_1, \ldots, Y_n$ and the δ-method.)

3. Let X be a continuous random variable with pdf

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta}) \exp\left[\sum_{i=1}^{k} \omega_i(\boldsymbol{\theta})t_i(x)\right],$$

where $h(x) \geq 0$, $t_1(x), \cdots, t_k(x) \in \mathbb{R}$, $c(\boldsymbol{\theta}) \geq 0$ and $\omega_1(\boldsymbol{\theta}), \cdots, \omega_k(\boldsymbol{\theta}) \in \mathbb{R}$. Show the following:

(a)

$$\mathrm{E}\left[\sum_{i=1}^{k} \frac{\partial \omega_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right] = -\frac{\partial}{\partial \theta_j} \log c(\boldsymbol{\theta})$$

(b)

$$\mathrm{Var}\left[\sum_{i=1}^{k} \frac{\partial \omega_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right] = -\frac{\partial^2}{\partial \theta_j^2} \log c(\boldsymbol{\theta}) - \mathrm{E}\left[\sum_{i=1}^{k} \frac{\partial^2 \omega_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(X)\right]$$

4. Let $X_1, X_2, \ldots, X_n$ be iid with density $f_\theta(x) = \theta^{-1} \exp(-x/\theta)$ for $x > 0$, and denote the order statistics by $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$.

   (a) Show that $Y_1, \ldots, Y_n$ are iid with density f_θ, where $Y_i = (n - i + 1)[X_{(i)} - X_{(i-1)}]$ and $X_{(0)} \equiv 0$. You may assume this result for part (b).

   (b) Suppose you only get to observe $X_{(1)}, \ldots, X_{(k)}$, where k is a known constant. Starting from the Neyman-Pearson Lemma, derive the UMP test for $H_0 : \theta \leq 1$ vs. $H_1 : \theta > 1$. What common distribution tables can be used for finding the critical value of the test statistic? Justify your answer.